3.13 Find the potential in the infinite slot of Ex. 3.3 if the boundary at $x = 0$ consists of two metal strips: one from $y = 0$ to $y = a/2$, is held at a constant potential V_o, and the other, from $y = a/2$ to $y = a$, is held at potential $-V_o$.

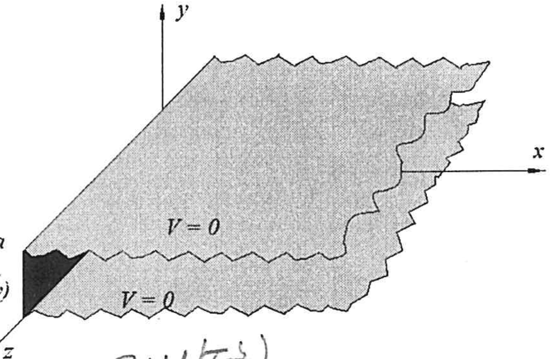


Figure 3.17

2 dimensions

The form of the solution is

$$V(x,y) = (C_1 e^{kx} + C_2 e^{-kx})(D_1 \cos ky + D_2 \sin ky)$$

Since $V \to 0$ at $x \to \infty$ $C_1 = 0$

Since $V = 0$ at $y = 0$ $D_1 = 0$

$$V(x,y) = C e^{-kx} \sin ky$$

Since $V = 0$ at $y = a$ $\sin ka = 0$ $ka = n\pi$

$k = n\pi/a$

$$V(x,y) = C_n e^{-\frac{n\pi}{a}x} \sin\left(\frac{n\pi}{a} y\right)$$

@ $x = 0$ $V(0,y) = \begin{cases} V_0 & 0 < y < a/2 \\ -V_0 & a/2 < y < a \end{cases}$

V, V_0, $-V_0$, y

$$\int_0^a V(0,y) \sin \frac{m\pi}{a} y \, dy = \sum_n C_n \int_0^a \sin \frac{m\pi}{a} y \sin \frac{n\pi}{a} y \, dy$$

$$= C_m \frac{a}{2}$$

$$C_m = \frac{2}{a} \int_0^a V(0,y) \sin\left(\frac{n\pi}{a} y\right) dy$$

$$C_n = \frac{2}{a} \int_0^{a/2} V_0 \sin \frac{n\pi}{a} y \, dy - \frac{2V_0}{a} \int_{a/2}^a \sin \frac{n\pi}{a} y \, dy.$$

3.13 CONTINUED

$$C_n = \frac{2V_0}{a}\left[\int_0^{a/2} \sin\frac{n\pi}{a}y\,dy - \int_{a/2}^{a} \sin\frac{n\pi}{a}y\,dy\right]$$

$$= \frac{2V_0}{a}\left[\left.\frac{-\cos\frac{n\pi}{a}}{\frac{n\pi}{a}}\right|_0^{a/2} + \left.\frac{\cos\frac{n\pi}{a}}{\frac{n\pi}{a}}\right|_{a/2}^{a}\right]$$

$$\cos\frac{n\pi}{a}\frac{a}{2} = \cos\frac{n\pi}{2}$$

$$\cos\frac{n\pi}{a}a = \cos n\pi$$

$$C_n = \frac{2V_0}{n\pi}\left[-\left(\cos\frac{n\pi}{2} - 1\right) + \left(\cos n\pi - \cos\frac{n\pi}{2}\right)\right]$$

$$= \frac{2V_0}{n\pi}\left[-2\cos\frac{n\pi}{2} + 1 + \cos n\pi\right]$$

n	$\cos\frac{n\pi}{2}$	$\cos n\pi$	$\left(-2\cos\frac{n\pi}{2} + 1 + \cos n\pi\right)$
0	1	1	0
1	0	-1	0
2	-1	1	4
3	0	-1	0
4	1	1	0
5	0	-1	0
6	-1	1	4

$$C_n = \frac{8V_0}{n\pi} \qquad n = 2, 6, 10, 14 \ldots$$

3.13 CONTINUED (PAGE 3)

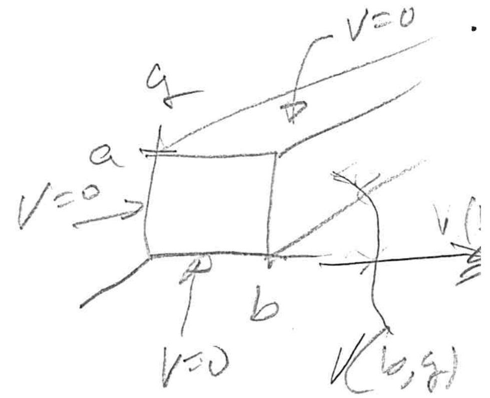
$$V(x,y) = \sum C_n e^{-\frac{n\pi}{a}x} \sin\frac{n\pi}{a}y$$

$$V(x,y) = \frac{8V_0}{\pi} \sum_{n=2,6,10,14,\ldots} \frac{1}{n} e^{-\frac{n\pi}{a}x} \sin\frac{n\pi}{a}y$$

3.15 A rectangular pipe, running parallel to the *z* axis (from minus infinity to plus infinity), has three grounded metal sides, at *y = 0, y = a,* and *x = 0*. The fourth side, at *x = b*, is maintained at a specific potential $V_o(y)$.

a. Develop a general formula for the potential inside the pipe.

b. Find the potential explicitly, for the case $V_o(y) = V_o$ (a constant).



$V = 0$ $\quad x = 0$
$\quad\quad\quad y = 0$
$\quad\quad\quad y = a$

The form of the solution is

$$V(x,y) = (A_1 e^{kx} + A_2 e^{-kx})(B_1 \sin ky + B_2 \cos ky)$$

OR $V(x,y) = (A_1 \sinh kx + A_2 \cosh kx)(B \sin ky + B_2 \cos ky)$

Since $V = 0$ at $x = 0$ $\quad A_2 = 0$
$V = 0$ at $y = 0$ $\quad B_2 = 0$
$V = 0$ at $y = a \to \sin ka = 0 \quad ka = n\pi$
$$k = \frac{n\pi}{a}$$

$$V(x,y) = c \sinh\left(\frac{n\pi}{a}x\right) \sin\left(\frac{n\pi}{a}y\right)$$

$$V(x,y) = \sum C_n \sinh\left(\frac{n\pi}{a}x\right) \sin\left(\frac{n\pi}{a}y\right)$$

at $x = b$ $\quad V(b,y) = \underbrace{C \sinh\left(\frac{n\pi}{a}b\right)}_{\text{CONSTANT}} \sin \frac{n\pi}{a} y$

$$V_0 = V(b,y) = \sum_n C_n \sinh\left(\frac{n\pi}{a}b\right) \sin \frac{n\pi}{a} y$$

$$\int_0^a V_0 \sin\left(\frac{m\pi}{a}y\right) dy = \sum C_n \sinh\left(\frac{n\pi}{a}b\right) \sin \frac{m\pi}{a}y \sin \frac{n\pi}{a}y$$
$$= C_n \sinh\left(\frac{n\pi}{a}b\right) \frac{a}{2} \quad n = m$$

3.15 PAGE 2 $\quad n = m$

$$\int_0^a V_0 \sin\left(\frac{n\pi}{a} y\right) dy = C_n \sin\left(\frac{n\pi b}{a}\right) \frac{a}{2}$$

$$\int_0^a V_0 \sin\left(\frac{n\pi}{a} y\right) dy = V_0 \left[\frac{-\cos\frac{n\pi}{a}y}{\frac{n\pi}{a}}\right]_0^a$$

$$= \frac{V_0 a}{n\pi}\left[1 - (-1)\right]$$

$$= \frac{2V_0 a}{n\pi}$$

$$= C_n \sin\left(\frac{n\pi}{a} b\right) \frac{a}{2}$$

$$\frac{2V_0 a}{n\pi} = C_n \sin\left(\frac{n\pi b}{a}\right) \frac{a}{2}$$

n odd

$$C_n = \frac{4V_0}{n\pi \sinh\left(\frac{n\pi b}{a}\right)} \qquad C_n = 0 \quad n \text{ even}$$

$$V(x,y) = \sum C_n \sinh\left(\frac{n\pi}{a} x\right) \sin\left(\frac{n\pi}{a} y\right)$$

$$V(x,y) = \frac{4V_0}{\pi} \sum_{n \text{ odd}} \frac{\sinh\left(\frac{n\pi}{a} x\right) \sin\left(\frac{n\pi}{a} y\right)}{n \sinh\left(\frac{n\pi}{a} b\right)}$$

3.16 A cubical box (sides of length *a*) consists of five metal plates, which are welded together and grounded (Fig 3.23). The top is made of a separate sheet of medal, insulated from the others, and held at a constant potential V_o. Find the potential inside the box. [What should the potential be at the center ($a/2$, $a/2$, $a/2$) be? Check numerically that your formula is consistent with this value.]

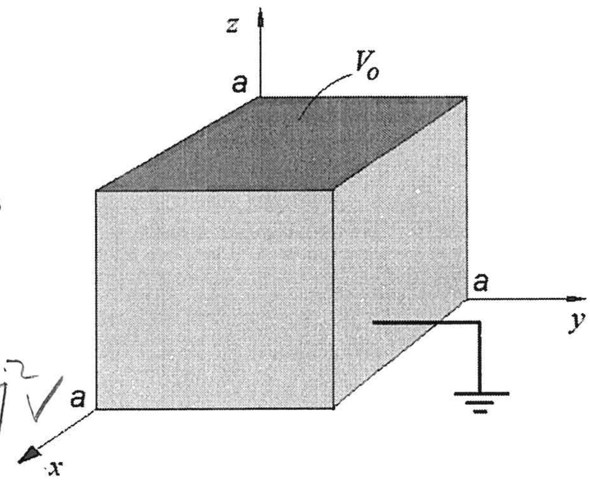


Figure 3.23

$V(x,y,z) = X(x)\,Y(y)\,Z(z)$

Since $\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0 = \nabla^2 V$

$$\frac{1}{X}\frac{d^2X}{dx^2} + \frac{1}{Y}\frac{d^2Y}{dy^2} + \frac{1}{Z}\frac{\partial^2 Z}{\partial z^2} = 0$$

$\frac{1}{X}\frac{\partial^2 X}{\partial x^2} = -k^2$ $\quad X(x) = A_1 \sin(kx) + A_2 \cos(kx)$

$\frac{1}{Y}\frac{\partial^2 Y}{\partial y^2} = -l^2$ $\quad Y(y) = B_1 \sin(ly) + B_2 \cos(ly)$

$\frac{1}{Z}\frac{\partial^2 Z}{\partial z^2} = g^2$ $\quad Z(z) = C_1 e^{gz} + C_2 e^{-gz}$

Since $\nabla^2 V = 0$ $\quad -k^2 - l^2 + g^2 = 0$

$g = \sqrt{k^2 + l^2}$

Boundary Conditions

@ $x = 0$; $V = 0$ $\therefore$ $A_2 = 0$

@ $y = 0$, $V = 0$ $\therefore$ $B_2 = 0$

@ $z = 0$, $V = 0$ $\therefore$ $C_1 = -C_2$

$Z(z) = C_1 (e^{gz} - e^{gz})$

$= 2 \sinh gz$

$V(x,y,z) = C_n \sin kx \sin ly \sinh gz$

3.16 ~~PAGE~~ 2

Since $V=0$ at $x=a$ $\quad \sin ka = 0$

$V=0$ at $y=a$ $\quad k = \frac{n\pi}{a}$

$\sin(l\, a) = 0 \quad l = \frac{m\pi}{a}$

$$V(x,y,z) = C_{n,m} \sin\frac{n\pi}{a}x \ \sin\frac{m\pi}{a}y \ \sinh(g z)$$

$$g^2 = \sqrt{\left(\frac{n\pi}{a}\right)^2 + \left(\frac{m\pi}{b}\right)^2}^{\,2}$$

We need a Sum of these solutions to satisfy the boundary condition at $z=a$

$$V(x,y,z) = \sum_n \sum_m C_{nm} \sin\frac{n\pi}{a}x \ \sin\frac{m\pi}{a}y \ \sinh g z$$

at $z=a$

$$V(x,y,a) = V_0 = \sum_n \sum_m C_{nm} \sin\frac{n\pi}{a}x \ \sin\frac{m\pi}{a}y \ \sinh(g a)$$

Use Fourier Series

$$V_0 = \sum_n \sum_m B_{nm} \sin\frac{n\pi}{a}x \ \sin\frac{m\pi}{a}x$$

where $B_{nm} = C_{nm} \sinh(g a)$

$$g = \sqrt{\left(\frac{n\pi}{a}\right)^2 + \left(\frac{m\pi}{a}\right)^2}^{\,2}$$

$$= \frac{\pi}{a}\sqrt{n^2+m^2}$$

$$V_0 = \sum_n \sum_m B_{nm} \sin\frac{n\pi}{a}x \sin\frac{m\pi}{a}y$$

$$\int_0^a V_0 \sin\frac{n'\pi}{a}x \sin\frac{m'\pi}{a}y \, dx\,dy =$$

$$\sum B_{nm} \int_0^a \int_0^a \sin\frac{n'\pi}{a}x \sin\frac{n\pi}{a}x \; * \; \sin\frac{m'\pi}{a}y \sin\frac{m\pi}{a}y \, dx$$

$$\int_0^a \sin\frac{n'\pi}{a}x \sin\frac{n\pi}{a}x\,dx = \frac{a}{2} \quad \text{if } n=n' \qquad = 0 \quad n \neq n'$$

$$\int_0^a \int_0^a V_0 \sin\frac{n'\pi}{a}x \sin\frac{m'\pi}{a}y \, dx\,dy = \frac{a^2}{4} B_{n',m'}$$

$$V_0 \int_0^a \int_0^a \sin\frac{n\pi}{a}x \sin\frac{m\pi}{a}y \, dx\,dy = V_0 \int_0^a \sin\frac{n\pi}{a}x\,dx \; * \; \int_0^a \sin\frac{m\pi}{a}y\,dy$$

$$= V_0 \left[-\frac{\cos\frac{n\pi}{a}x}{\frac{n\pi}{a}} \Big|_0^a \right] \left[-\frac{\cos\frac{m\pi}{a}y}{\frac{m\pi}{a}} \Big|_0^a \right]$$

$$= \frac{V_0 4 a^2}{nm\pi^2} \quad \text{odd } n \text{ and odd } m$$

(3.16 PAGE 4)

$$\frac{V_0 4 \cancel{a^2}}{nm\pi^2} = \frac{\cancel{a^2}}{4} B_{n,m}$$

$$B_{nm} = \frac{16 V_0}{nm\pi^2} = C_{nm} \sinh(\gamma a)$$

$$V(x,y,z) = \sum_{\text{odd } n} \sum_{\text{odd } m} C_{n,m} \sin\frac{n\pi}{a}x \sin\frac{m\pi}{a}y \sinh \gamma z$$

where $\gamma = \frac{\pi}{a}\sqrt{n^2+m^2} = \sqrt{k^2+l^2}$

$$V(x,y,z) = \frac{16V_0}{\pi^2} \sum_{\text{odd } n} \sum_{\text{odd } m} \frac{\sin\frac{n\pi}{a}x \sin\frac{m\pi}{a}y \sinh\left(\frac{\pi}{a}\sqrt{n^2+m^2}\,z\right)}{nm \sinh\left(\pi\sqrt{n^2+m^2}\right)}$$