2.21 Find the potential inside and outside a uniformly charged solid sphere whose radius is R and whose charge is q. Use infinity as your reference point. Compute the gradient of V in each region, and check that it yields the correct field.

Find $\vec{E}$ then calculate $V(r) = -\int_{\infty}^{r} E_r\, dr$

$\boxed{r > R}$ $\vec{E} = \frac{Q\,\hat{r}}{4\pi\epsilon_0 r^2}$ + $V = \frac{Q}{4\pi r}$ Same as a point charge at the center

$\boxed{r < R}$ $\oint \vec{E}\cdot d\vec{a} = \frac{Q_{enc}}{\epsilon_0} = 4\pi r^2 E_r$

$$E_r = \frac{Q_{enc}}{4\pi\epsilon_0 r^2}$$

$$Q_{enc} = \rho \frac{4}{3}\pi r^3 \qquad \rho = \frac{Q}{\frac{4}{3}\pi R^3}$$

$$Q_{enc} = \left(\frac{r}{R}\right)^3 Q$$

$$E_r = \frac{Q\, r}{4\pi\epsilon_0 R^3}$$

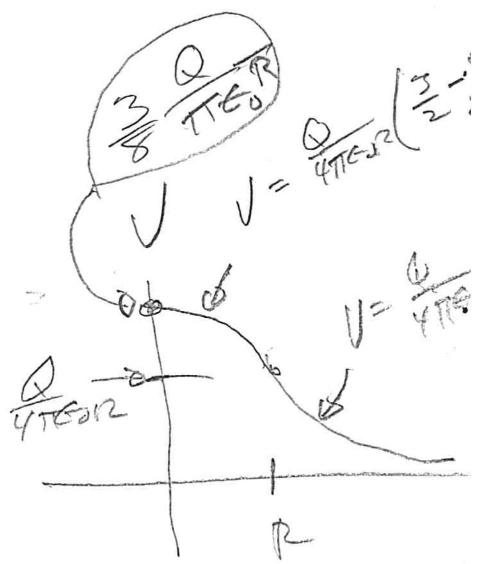
$$V(r) = -\int_{\infty}^{r} E_r\, dr = \underbrace{-\int_{-\infty}^{R} E_r\, dr}_{\substack{V(R) \\ \frac{Q}{4\pi\epsilon_0 R}}} - \int_{R}^{r} E_r\, dr$$

$$V(r) = \frac{Q}{4\pi\epsilon_0 R} - \int_{R}^{r} \frac{Q\, r\, dr}{4\pi\epsilon_0 R^3}$$

$$V(r) = \frac{Q}{4\pi\epsilon_0 R} - \frac{Q}{4\pi\epsilon_0 R^3}\left[\frac{r^2}{2} - \frac{R^2}{2}\right] = \frac{Q}{4\pi\epsilon_0 R}\left[1 + \frac{1}{2} - \frac{r^2}{2R^2}\right]$$

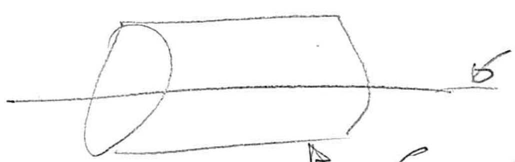
$$V(r) = \frac{Q}{4\pi\epsilon_0 R}\left[\frac{3}{2} - \frac{r^2}{2R^2}\right] \qquad 0 < r < R \qquad \nabla V = \frac{\partial V}{\partial r}$$

$$\nabla V = \frac{\partial V}{\partial r} = -\frac{Q}{4\pi\epsilon_0}\frac{r}{R^3} \quad -E \qquad \boxed{\nabla V = -E}$$



2.22 Find the potential a distance *s* from an infinitely long straight wire that carries a uniform line charge . Compute the gradient of your potential, and check that it yields the correct field.

FIRST FIND E then use $V(r)-V(a) = -\int_a^r E_r dr$



$$\frac{Q_{enc}}{\epsilon_0} = \int \vec{E}\cdot d\vec{a} = L\,2\pi r E_r = \frac{\lambda L}{\epsilon_0}$$

$$E_r = \frac{\lambda}{2\pi\epsilon_0 r}$$

$$V(r) - V(a) = -\int_a^r E_r dr = -\frac{\lambda}{2\pi\epsilon_0}\ln\left(\frac{r}{a}\right)$$

$$\boxed{\begin{array}{l}\text{Let } V(a)=0 \\ V(r) = -\frac{\lambda}{2\pi\epsilon_0}\ln(r/a)\end{array}}$$

$$\frac{d}{dx}(\ln x) = \frac{1}{x}$$

$$\frac{d}{dx}(\ln ax) = \frac{1}{ax}$$

$$\vec{\nabla} V = \frac{\partial V}{\partial r}\hat{r}$$

$$\frac{\partial}{\partial r}\left(\frac{-\lambda}{2\pi\epsilon_0}\ln\frac{r}{a}\right) = -\frac{\lambda}{2\pi\epsilon_0} * \left(\frac{a}{r}\right) * \frac{1}{a}$$

* chain rule

$$\boxed{\vec{\nabla} V = -\frac{\lambda}{2\pi\epsilon_0}\frac{\hat{r}}{r} = -E_r\hat{r}}$$

2.31

a. Three charges are situated at the corners of a square (side *a*), as shown in Fig. 2.41. How much work does it take to bring in another charge, +*q*, from far away and place it in the forth corner?

b. How muich work does it take to assemble the whole configuration of four charges?

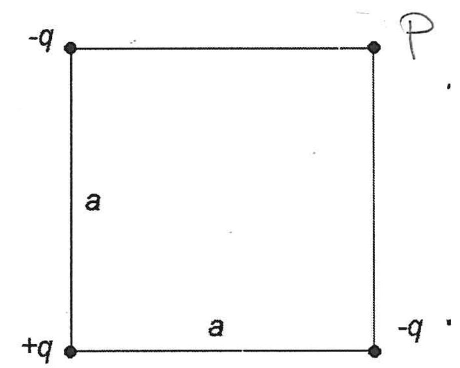


Figure 2.41

$$V(P) = \frac{-2q}{4\pi\epsilon_0 a} + \frac{q}{4\pi\epsilon_0 \sqrt{2}\, a}$$

$$= \frac{q}{4\pi\epsilon_0 a}\left[\frac{1}{\sqrt{2}} - 2\right] = \frac{q}{4\pi\epsilon_0 a}\left(\frac{2\sqrt{2}-1}{\sqrt{2}}\right)$$

a) $$W = q\,V(P) = \frac{q^2}{4\pi\epsilon_0 a}\left(\frac{2\sqrt{2}-1}{\sqrt{2}}\right)$$

b) $$W_{Total} = \frac{-q^2}{4\pi\epsilon_0 a} + \frac{-q^2}{4\pi\epsilon_0 a} + \frac{q^2}{4\pi\epsilon_0 \sqrt{2}\, a} + \frac{q^2}{4\pi\epsilon_0 a}\left(\frac{2\sqrt{2}}{\sqrt{2}}\right.$$

$$= \frac{q^2}{4\pi\epsilon_0 a}\left[-2 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} - 2\right]$$

$$W_{Total} = \frac{q^2}{4\pi\epsilon_0 a}\left[\frac{4\sqrt{2}-2}{\sqrt{2}}\right]$$

2.36 Consider two concentric spherical shells, of radii *a* and *b*. Suppose the inner one carries a charge *q*, and the outer one a charge -*q* (both of them uniformily distributed over the surface). Calculate the energy of this configuration, (a) using Eq. 2.45, and (b) using Eq. 2.47 and the results of Ex. 2.9.

$$W = \frac{\epsilon_o}{2}\int E^2 d\tau \qquad (2.45)$$

$$W_{tot} = W_1 + W_2 + \epsilon_o \int E_1 \cdot E_2 d\tau \qquad (2.47)$$

$$\vec{E} = \frac{q\,\hat{r}}{4\pi\epsilon_0 r^2} \qquad a < r < b$$

$$\frac{1}{2}\epsilon\int E^2 d\tau = \frac{\epsilon}{2}\int_a^b \left(\frac{q}{4\pi\epsilon_0}\right)^2 \frac{4\pi r^2 dr}{r^4}$$

$$= \frac{\epsilon}{2}\left(\frac{q}{4\pi\epsilon_0}\right)^2 4\pi \int_a^b \frac{dr}{r^2} = \frac{\epsilon_0}{2}\left(\frac{q}{4\pi\epsilon_0}\right)^2 4\pi \left[-\frac{1}{r}\right]$$

$$W = \frac{1}{2}\frac{q^2}{4\pi\epsilon_0}\left[\frac{1}{a} - \frac{1}{b}\right]$$

$$E = \begin{cases} \dfrac{q}{4\pi\epsilon_0 r} & a < r < b \\ 0 & \text{otherwise} \end{cases}$$

$$V = 0 \qquad r > a$$

2.36

From Example 2.9

For a spherical charged shell

$$W = \frac{1}{8\pi\epsilon_0}\left(\frac{q^2}{R}\right)$$

$$W_1 = \frac{q^2}{8\pi\epsilon_0}\left(\frac{1}{a}\right)$$

$$W_2 = \frac{q^2}{4\pi\epsilon_0}\left(\frac{1}{b}\right)$$

$$E_1 = \frac{q}{4\pi\epsilon_0 r^2} \quad r > a$$

$$\epsilon_0 \int E_1 \cdot E_2 \, d\tau$$

$$E_2 = \frac{-q}{4\pi\epsilon_0 r^2} \quad r > b$$

$$E_1 \cdot E_2 = -\left(\frac{q^2}{4\pi\epsilon_0}\right)^2 \frac{1}{r^4}$$

$$\epsilon_0 \int E_1 \cdot E_2 \, d\tau = -\epsilon_0 \int_b^\infty \left(\frac{q}{4\pi\epsilon_0}\right)^2 \frac{4\pi r^2}{r^4} dr$$

$$= \frac{-q^2}{4\pi\epsilon_0}\left[\frac{r^{-1}}{-1}\right]_b^\infty = \frac{-q^2}{4\pi\epsilon_0 b}$$

$$W_1 + W_2 + \epsilon_0 \int E_1 \cdot E_2 \, d\tau =$$

$$W = \frac{q^2}{8\pi\epsilon_0}\left[\frac{1}{a} + \frac{1}{b} - \frac{2}{b}\right] = \frac{q^2}{8\pi\epsilon_0}\left[\frac{1}{a} - \frac{1}{b}\right].$$