2.3 Find the electric field a distance z above one end of a straight line segment of length L. (Fig 2.7) that carries a uniform line charge λ. Check that your formula is consistent with what you would expect for the case $z >> L$.

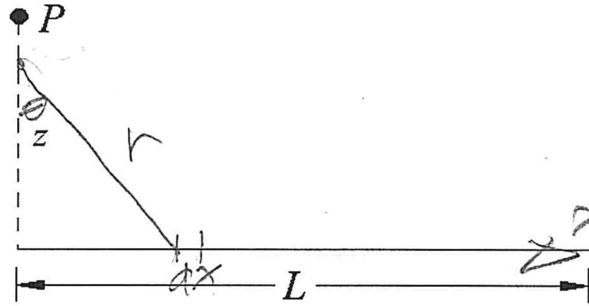


Figure 2.7

$$\frac{x}{z} = \tan\theta$$

$$x = z\tan\theta$$

$$dx = z\sec^2\theta\, d\theta$$

$$\frac{z}{r} = \cos\theta$$

$$\frac{1}{r^2} = \frac{\cos^2\theta}{z^2}$$

$$dq = \lambda dx$$

$$\vec{E} = \frac{1}{4\pi\epsilon_0}\int \frac{dq\,\hat{r}}{r^2}$$

$$E_z = E\cos\theta = \frac{1}{4\pi\epsilon_0}\int \frac{\lambda dx\cos\theta}{r^2}$$

$$= \frac{1}{4\pi\epsilon_0}\int \lambda \cancel{z}\sec^2\theta\, d\theta\, \cos\theta\, \frac{\cos^2\theta}{z^{\cancel{2}}}$$

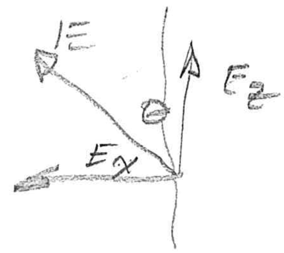
$$= \frac{\lambda}{4\pi\epsilon_0 z}\int_0^{\theta_L} \cos\theta\, d\theta$$

$$E_z = \frac{\lambda}{4\pi\epsilon_0 z}\Big[\sin\theta\Big]_0^{\theta_L} = \frac{\lambda \sin\theta_L}{4\pi\epsilon_0 z}$$

$$\sin\theta_L = \frac{L}{r} = \frac{L}{\sqrt{L^2+z^2}}$$

$$E_z = \frac{\lambda L}{4\pi\epsilon_0 z\sqrt{L^2+z^2}}$$

2.3 Continued



$$E_x = -E\sin\theta$$

$$= \frac{-\lambda}{4\pi\epsilon_0 z}\int_0^{\theta_L}\sin\theta\,d\theta$$

$$= \frac{-\lambda}{4\pi\epsilon_0 z}\left[-\cos\theta\Big|_0^{\theta_L}\right.$$

$$= \frac{\lambda}{4\pi\epsilon_0 z}\left[\cos\theta_L - 1\right]$$

E_z is proportional to $\cos\theta$

E_x is proportional to $-\sin\theta$

$$\cos\theta_L = \frac{z}{r} = \frac{z}{\sqrt{z^2+L^2}}$$

$$\vec{E} = E_x\hat{x} + E_z\hat{z}$$

$$= \frac{\lambda}{4\pi\epsilon_0 z}\left(\left[\frac{z}{\sqrt{z^2+L^2}} - 1\right]\hat{x} + \frac{L\hat{y}}{\sqrt{z^2+L^2}}\right)$$

2.4 Find the electric field a distance *z* above the center of a square loop (side a) that carries a uniform line charge λ (Fig 2.8). [*Hint:* Use the results of Ex. 2.2.]

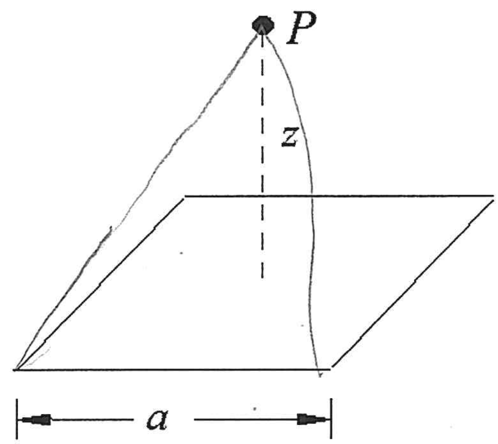
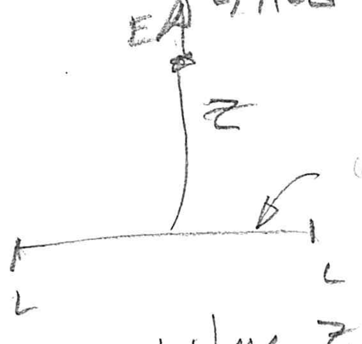


Figure 2.8

Ex 2.2 RESULT

$$\vec{E} = \frac{1}{4\pi\epsilon_0}\frac{2\lambda L\, z}{\sqrt{z^2+L^2}}$$

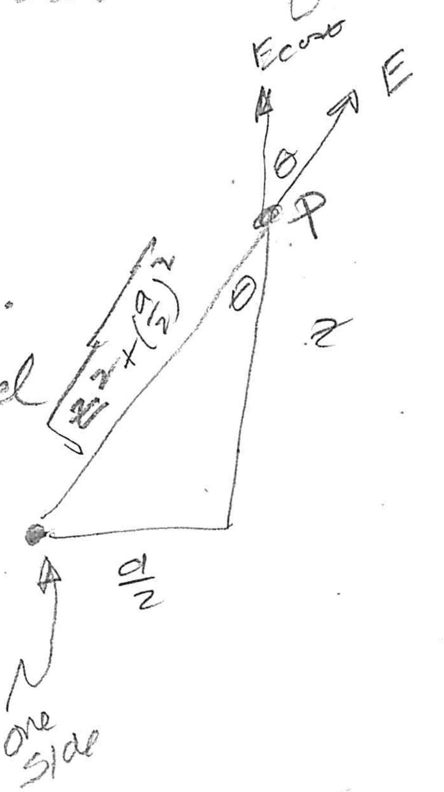


where z is the distance from the wire

In this problem the distance from the wire is $\sqrt{z^2+(\frac{a}{2})^2}$

The vertical components of the E field due to the 4 sides add. The horizontal components cancel

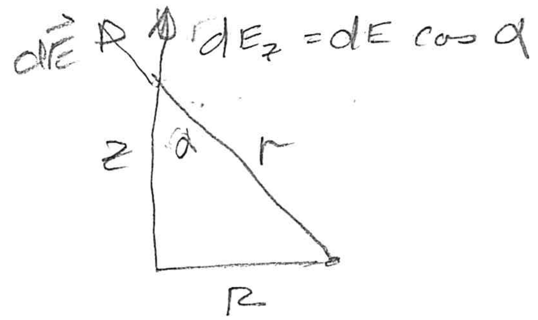
$$\vec{E} = 4E_z = 4\cos\theta\,\frac{2\lambda\, a/2}{\sqrt{z^2+(\frac{a}{2})^2+(\frac{a}{2})^2}}$$



$$E = \frac{1}{4\pi\epsilon_0} * \frac{4\lambda a z}{\sqrt{\left(z^2+2(\frac{a}{2})^2\right)}\left(z^2+(\frac{a}{2})^2\right)}$$

$$\cos\theta = \frac{z}{\sqrt{z^2+(\frac{a}{2})^2}}$$

2.5 Find the electric field a distance *z* above the center of a circular loop of radius *r* (Fig. 2.9) that carries a uniform line charge λ.



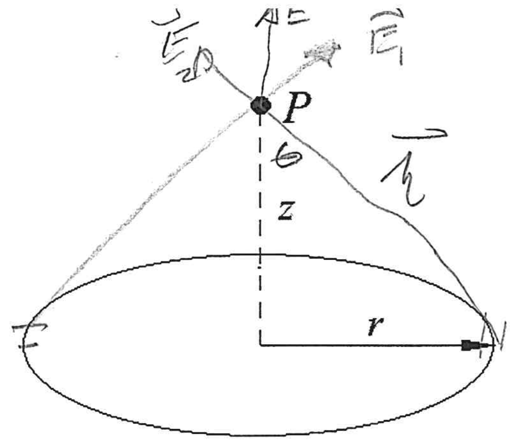


Figure 2.9

$$\cos\alpha = \frac{z}{\mathscr{r}}$$

$$d\vec{E} = \frac{dq}{4\pi\epsilon_0 \mathscr{r}^2}\hat{\mathscr{r}}$$

$$dE_z = \frac{dq\cos\alpha}{4\pi\epsilon_0 \mathscr{r}^2}$$

$$\mathscr{r}^2 = r^2 + z^2$$

$$dq = \lambda\, dl = \lambda r\, d\theta$$

$$dE_z = \frac{\lambda r\, d\theta \cos\alpha}{4\pi\epsilon_0 (r^2+z^2)} = \frac{\lambda r\, d\theta\, z}{4\pi\epsilon_0 (r^2+z^2)^{3/2}}$$

$$E_z = \int dE_z = \int_0^{2\pi} \frac{\lambda r z \, d\theta}{4\pi\epsilon_0 (r^2+z^2)^{3/2}}$$

$$\vec{E} = \hat{z} E_z = \frac{\lambda r z\, \hat{z}}{2\epsilon_0 (r^2+z^2)^{3/2}}$$

2.9 Suppose the electric field in some region is found to be $\mathbf{E} = k r^3 \hat{\mathbf{r}}$, in spherical coordinates (k is some constant.)

a. Find the charge density ρ.

b. Find the total charge contained in a sphere of radius R, centered at the origin. (Do it in two different ways.)

$$\nabla \cdot E = \rho / \epsilon_0$$

NOT FUNCTIONS
θ or φ

$$\rho = \epsilon_0 \nabla \cdot E$$

$$\vec{\nabla} \cdot \vec{E} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 E_r) + \cancel{\frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta E_\theta)}^{\,0} + \cancel{\frac{1}{r\sin\theta}\frac{\partial E_\phi}{\partial\phi}}^{\,0}$$

$$\rho = \epsilon_0 \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 E_r)$$

$$= \epsilon_0 \frac{1}{r^2}\frac{\partial}{\partial r}(k r^5) = \epsilon_0 \frac{1}{r^2}(5 k r^4)$$

a) $\boxed{\rho = \epsilon_0 5 k r^2}$

$$\oint \vec{E} \cdot d\vec{a} = \frac{Q_{enc}}{\epsilon_0}$$

$$\oint \vec{E} \cdot da = E_r 4\pi R^2 = \frac{Q_{enc}}{\epsilon_0}$$

$$Q_{enc} = \epsilon_0 4\pi r^2 E_r = \epsilon_0 4\pi r^2 k R^3$$

$$Q_{enc} = 4\pi\epsilon_0 k R^5$$

Method #2 $\quad Q_{enc} = \int \rho \, dv \quad$ $dv = 4\pi r^2 dr$

$$= \int_0^R \epsilon_0 5 k r^2 \, 4\pi r^2 dr = 20\pi k \epsilon_0 \int_0^R r^4 dr$$

$$= \frac{20\pi\epsilon_0 k R^5}{5} = 4\pi\epsilon_0 k R^5$$