1.27 Prove that the divergence of the curl is always zero. *Check* it for the function $\mathbf{V}_a$ in Problem 1.15.

$$\vec{v_a} = x^2\hat{x} + 3xz^2\hat{y} - 2xz\hat{z}$$

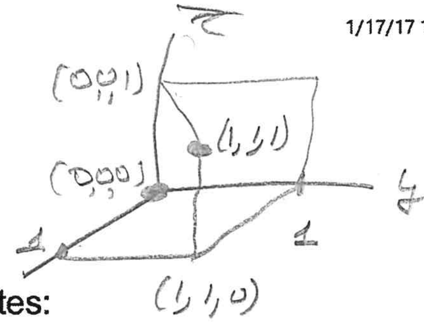
$$\vec{\nabla} \times \vec{v_a} = \left| \begin{matrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 & 3xz^2 & -2xz \end{matrix} \right| \begin{matrix} \hat{x} & \hat{y} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ x^2 & 3xz^2 \end{matrix}$$

$$\vec{\nabla \times v_a} = \hat{x}\left[-6xz\right] + \hat{y}\, z + \hat{z}\left[3z^2\right]$$

$$= \vec{C}$$

$$\vec{\nabla} \cdot \vec{C} = \frac{\partial C_x}{\partial x} + \frac{\partial C_y}{\partial y} + \frac{\partial C_z}{\partial z} = \nabla \cdot \vec{\nabla \times v_a}$$

$$= -6z + 6z = 0$$



1.29 Calculate the line integral of the function;

$$\vec{v} = x^2\hat{x} + 2yz\hat{y} + y^2\hat{z}$$

from the origin to the point (1, 1, 1) by means of three different routes:

a. (0, 0, 0) --> (1, 1, 0) --> (1, 1, 1)

b. (0, 0, 0) --> (0, 0, 1) --> (1, 1, 1)

c. The direct straighht line.

d. What is the line integral around the closed loop that goes *out* along path (a) and *back* along path (b)?

$$\int_{0,0,0}^{1,1,0} \vec{v}\cdot d\vec{l} = \int_0^1 v_x\,dx + \int_0^1 v_y\,dy$$

where $x = y$ $dx = dy$

$$\int_{0,0,0}^{1,1,0} \vec{v}\cdot d\vec{l} = \int_0^1 x^2\,dx + \int_0^1 2y\cancel{z}^{\,0}\,dy \quad z=0 \quad = \frac{x^3}{3}\Big|_0^1 = \frac{1}{3}$$

$$\int_{1,1,0}^{1,1,1} \vec{v}\cdot d\vec{l} = \int_0^1 v_z\,dz = \int_0^1 y^2\,dz = 1$$

$x = 1$
$y = 1$

a) $\displaystyle\int_{0,0,0}^{(1,1,1)} v\cdot dl = \frac{4}{3}$

1.29

b) $$\int_{0,0,0}^{0,0,1} v \cdot dl = \int_0^1 v_z \, dz = \int_0^1 y^2 dz = 0$$

$x = 0$
$y = 0$

$$\int_{0,0,1}^{1,1,1} \vec{v} \cdot d\vec{l} = \int_0^1 v_x \, dx + \int_0^1 v_y \, dy$$

$x = y$ $\quad v_x = x^2$
$dx = dy$ $\quad v_y = 2yz$
$z = 1$ $\quad = 2y$

$$\int_{0,0,1}^{1,1,1} \vec{v} \cdot d\vec{l} = \int_0^1 x^2 dx + \int_0^1 2y \, dy$$

$$= \frac{x^3}{3}\Big|_0^1 + \frac{2y^2}{2}\Big|_0^1 = \frac{1}{3} + 1 = \frac{4}{3}$$

c) The direct Line $x = y = z$ $\quad dx = dy = dz$

$$\int v \cdot dl = \int_0^1 v_x \, dx + \int_0^1 v_y \, dy + \int_0^1 v_z \, dz$$

$$= \int_0^1 x^2 dx + \int_0^1 2yz \, dy + \int_0^1 y^2 dz$$

d) Zero

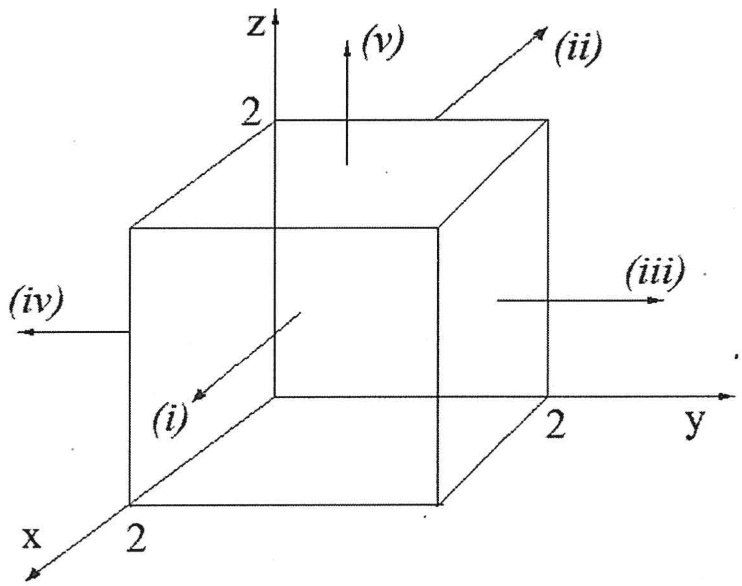
$\frac{4}{3} - \frac{4}{3} = 0$

$$= \int_0^1 x^2 dx + 2\int_0^1 y^2 dy + \int_0^1 z^2 dz$$

$$= \frac{1}{3} + \frac{2}{3} + \frac{1}{3} = \frac{4}{3}$$

1.30 Calculate the surface integral of the function in Ex 1.7 over the *bottom* of the box. For consistency, let "upward" be the positive direction. Does the surface integral depend only on the boundary line for this function? What is the total flux over the *closed* surface of the box (*including* the bottom)? [*Note:* For the *closed* surface, the positive direction is "outward", and hence "down", for the bottom face.]

$$\vec{v} = 2xz\,\hat{x} + (x+2)\,\hat{y} + y(z^2-3)\,\hat{z}$$



$\int_{\text{Bottom}} v \cdot da$

$z = 0$

$v_z = y(z^2-3) = -3y$

$\vec{da} = dx\,dy\,\hat{z}$

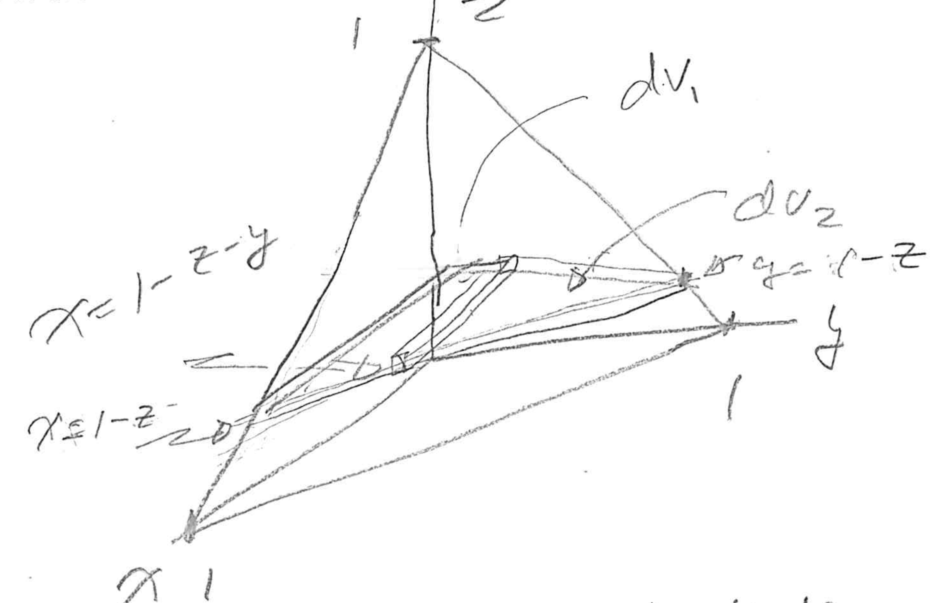
$$\int_{\text{Bottom}} \vec{v}\cdot\vec{da} = \int_0^2\int_0^2 -3y\,dx\,dy = \int_0^2\left[-3\frac{y^2}{2}\Big|_0^2\right]dx$$

$$= -\frac{3}{2}\,4\cdot\int_0^2 dx = -\frac{3}{2}\times 4\times 2 = \boxed{-12}$$

From Ex 1.7 the flux thru all the surface except the Bottom is 20

The Total flux $= 20 + 12 = \boxed{32}$

1.31 Calculate the volume integral of the function $T = z^2$ over the tetrahedron with corners at (0, 0, 0), (1, 0, 0), (0, 1, 0) and (0, 0, 1).



$dV = dx\,dy\,dz$

INTEGRATE OVER x

$$\iiint z^2\,dx\,dy\,dz$$

$dV_1 = (1-z-y)\,dy\,dz$

Integrate over y

$dV_2 = \frac{(1-z)^2}{2}\,dz$

dV_2 is the horizontal slice at z

$$\int_0^1 z^2 \int_0^{1-z} \int_0^{1-z-y} dx\,dy\,dz$$

$$\int_0^1 z^2 \int_0^{1-z} dy\,(1-z-y)\,dy\,dz$$

$$\int_0^1 z^2 \left[(1-z)y - \frac{y^2}{2}\right]_0^{1-z} dz = \int_0^1 z^2 (1-z)^2 - \frac{(1-z)^2}{2}\,dz$$

$$= \int_0^1 \frac{z^2(1-z)^2}{2}\,dz = \int_0^1 \frac{z^2 - 2z^3 + z^4}{2}\,dz = \frac{1}{2}\left[\frac{z^3}{3} - \frac{2z^4}{4} + \frac{z^5}{5}\right]_0^1$$

$$= \frac{1}{2}\left[\frac{1}{3} - \frac{1}{2} + \frac{1}{5}\right] = \frac{1}{2}\left[\frac{10-15+6}{30}\right] = \boxed{\frac{1}{60}}$$