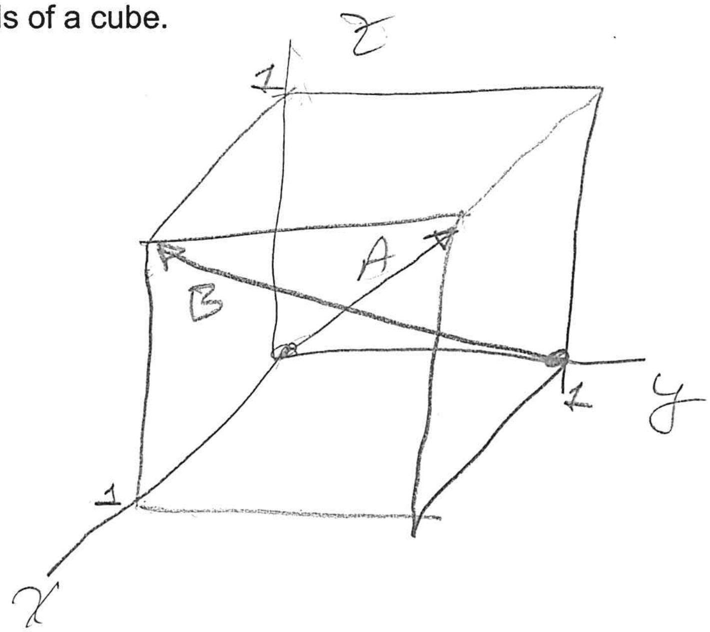
1.3 Find the angle between the body diagonals of a cube.



Unit cube

Diagonal #1

$\vec{A} = \hat{x} + \hat{y} + \hat{z}$

Diagonal #2

$\vec{B} = \hat{x} - \hat{y} + \hat{z}$

$|A| = |B| = \sqrt{1+1+1} = \sqrt{3}$

$\vec{A} \cdot \vec{B} = 1 - 1 + 1 = 1 = AB\cos\theta = \sqrt{3}\sqrt{3}\cos\theta$

$\cos\theta = \frac{1}{3}$

$\theta = \cos^{-1}\left(\frac{1}{3}\right) = 70.5^{\circ}$

1.4 Use the cross product to find the components of the unit vector $\hat{n}$ perpendicular to the shaded plane in Fig 1.11.

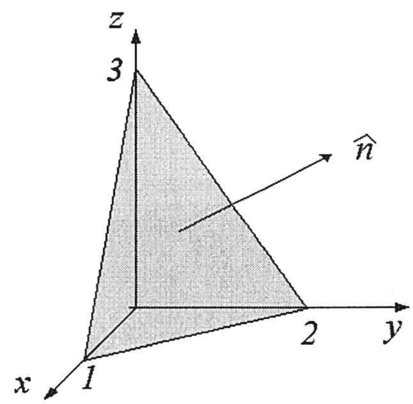


Figure 1.11

$\vec{A} \times \vec{B} = \vec{C}$

$\vec{C}$ is perpendicular to both $\vec{A}$ & $\vec{B}$

Let $\vec{A}$ & $\vec{B}$ lie in the plane

$\vec{A} = -\hat{x} + 2\hat{y}$ (base)

$\vec{B} = -2\hat{y} + 3\hat{z}$ (diagonal from $y=2$ to $z=3$)

$$\vec{A} \times \vec{B} = \vec{C} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ -1 & 2 & 0 \\ 0 & -2 & 3 \end{vmatrix} = 6\hat{x} + 3\hat{y} + 2\hat{z}$$

$$\hat{n} = \frac{\vec{C}}{C} = \frac{6\hat{x} + 3\hat{y} + 2\hat{z}}{\sqrt{36 + 9 + 4}}$$

$$\boxed{\hat{n} = \frac{6}{7}\hat{x} + \frac{3}{7}\hat{y} + \frac{2}{7}\hat{z}}$$

1.11 Find the gradients of the following functions:

a. $f(x, y, z) = x^2 + y^3 + z^4$

b. $f(x, y, z) = x^2 y^3 z^4$

c. $f(x, y, z) = e^x \sin(y) \ln(z)$

$$\vec{\nabla} f = \hat{x}\frac{\partial f}{\partial x} + \hat{y}\frac{\partial f}{\partial y} + \hat{z}\frac{\partial f}{\partial z}$$

a) $\frac{\partial f}{\partial x} = 2x \quad \frac{\partial f}{\partial y} = 3y^2 \quad \frac{\partial f}{\partial z} = 4z^3$

$$\vec{\nabla} f = 2x\,\hat{x} + 3y^2\,\hat{y} + 4z^3\,\hat{z}$$

b) $\frac{\partial f}{\partial x} = 2x y^3 z^4 \quad \frac{\partial f}{\partial y} = 3x^2 y^2 z^4 \quad \frac{\partial f}{\partial z} = 4x^2 y^3 z^3$

$$\vec{\nabla} f = 2xy^3 z^4\,\hat{x} + 3x^2 y^2 z^4\,\hat{y} + 4x^2 y^3 z^3\,\hat{z}$$

c) $\frac{\partial f}{\partial x} = e^x \sin(y) \ln(z)$

$\frac{\partial f}{\partial y} = e^x \cos(y) \ln(z)$

$\frac{\partial f}{\partial z} = \frac{e^x \sin(y)}{z}$

$$\vec{\nabla} f = e^x \sin(y) \ln(z)\,\hat{x} + e^x \cos(y) \ln(z)\,\hat{y} + \frac{e^x \sin(y)}{z}\,\hat{z}$$

Recall $\frac{\partial}{\partial z}(\ln(z)) = \frac{1}{z}$

1.12 The height of a certain hill (in feet) is given by

$$h(x, y) = 10(2xy - 3x^2 - 4y^2 - 18x + 28y + 12)$$

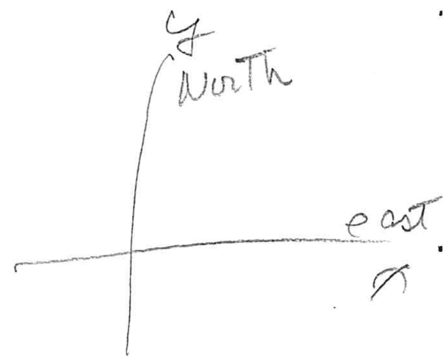
where *y* is the distance (in miles) north, *x* is the distance east of South Hadley.

a. Where is the top of the hill located?

b. How high is the hill?

c. How steep is the hill (in feet per mile) at a point 1 mile north and one mile east of South Hadley? In what direction is the slope steepest at that location?

$\text{Slope} = \vec{\nabla} h \cdot = \frac{\partial h}{\partial x}\hat{x} + \frac{\partial h}{\partial y}\hat{y}$



$\frac{\partial h}{\partial x} = 10(2y - 6x - 18)$

$\frac{\partial h}{\partial y} = 10(2x - 8y + 28)$

at the top $\vec{\nabla} h = 0$

$-6x + 2y - 18 = 0 \quad (1)$

$2x - 8y + 28 = 0 \quad (2)$

Multiply equation (1) by 4 and add to eq (2)

$-24x + 8y - 72 = 0$

$2x + 8y + 28 = 0$

$-22x + 0 - 44 = 0$

$x = -2$

Plug into EQ 1

$12 + 2y - 18 = 0$

$2y = 6 \quad y = 3$

Top is at

$(x, y) = (-2, 3)$

3 miles North

2 miles West

Top at 2 miles West and 3 miles North of South Hadley

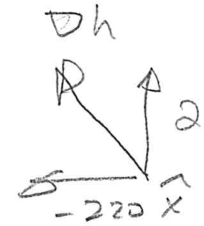
1.12 b How high is the hill at

$x = -2 \quad y = 3$

$h(-2,3) = 10(2*(-2)*3 - 3*4 - 4*9 - 18*(-2) + 28*3 - 12)$

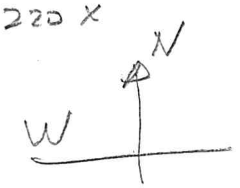
$= 10(12 - 12 - 36 + 36 + 84 - 12) \boxed{= 720 \text{ ft}}$

c) $\vec{\nabla h}(x,y) = 10[2y - 4x - 18]\hat{x} + 10[2x - 8y + 28]\hat{y}$



at $x = 1 \quad y = 1$

$\vec{\nabla h} = -220\hat{x} + 220\hat{y}$



Slope $= |\vec{\nabla h}| = \sqrt{2}\,220$

$= 310$ ft/mile

The direction is North west
45° North of West